The Expressive Power of the Modal μ-Calculus

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Introduction

"Started from the bottom now we here"

– Aubrey "Drake" Graham

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Transition systems model the behaviour of programs...
...hence it would be useful to state and verify properties of transition systems

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In 1983, Dexter Kozen introduced the modal μ-calculus $L\mu$, which enhances a simple syntax with powerful fixed-point operators and subsumes the logics above.

Today we will show that $L\mu$ subsumes PDL in particular. The goal is to show that $L\mu$ is **strictly** more expressive than PDL.

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Syntax of $L\mu$

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Atomic propositions $P \in AP$. Includes $\top$

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Greatest fixed point of φ

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Syntax of $L\mu$

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The other usual operators can be obtained by de Morgan duality:

$$\bot \equiv \neg\top$$

$$\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$$

$$\langle a\rangle\varphi \equiv \neg[a]\neg\varphi$$

$$\mu X.\varphi(X) \equiv \neg\nu X.\neg\varphi(\neg X)$$

$*$

Semantics of $L\mu$

We can define the semantics of $L\mu$ in terms of states of a transition system TS over a set of states S, where we have a function $D : AP \to 2^S$ mapping atomic propositions to the states at which they hold ($D(\top) = S$). We define $[\![\varphi]\!]$, the set of all states satisfying φ, inductively as follows:

$$[\![P]\!] = D(P)$$
$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$
$$[\![\neg\varphi]\!] = S \setminus [\![\varphi]\!]$$
$$[\![[a]\varphi]\!] = \{s \in S \,|\, \forall t \,.\, s \xrightarrow{a} t \implies t \in [\![\varphi]\!]\}$$
$$[\![\langle a\rangle\varphi]\!] = \{s \in S \,|\, \exists t \,.\, s \xrightarrow{a} t \wedge t \in [\![\varphi]\!]\}$$

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Semantics of $L\mu$

If a formula contains a variable X, we interpret $[\![\varphi(X)]\!]$ as a function $T \mapsto [\![\varphi[T/X]]\!]$ mapping sets of states $T \subseteq S$ to an interpretation of φ where all instances of X have been replaced by the states in T. We interpret this mixing of formulas and states like this (for example):

$$s \in [\![\psi \wedge T]\!] \text{ if } s \in [\![\psi]\!] \text{ and } s \in T$$

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For notational simplicity we will consider formulas of a single variable, and write $[\![\varphi(\psi)]\!]$ to express $[\![\varphi(X)]\!]([\![\psi]\!])$.

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Semantics of $L\mu$

Formulas $\varphi(X)$ that obey the positivity restriction define monotonic functions $[\![\varphi(X)]\!] : 2^S \to 2^S$ on the powerset lattice, which is complete. Hence we can define $[\![\mu X.\varphi(X)]\!]$ and $[\![\nu X.\varphi(X)]\!]$ to be the least and greatest fixed points of $[\![\varphi(X)]\!]$.

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Recursion semantics

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Furthermore, we may obtain these fixed points by successive iterations of f. For instance, $\mu f = \bigvee_n f^n(\bot)$

Hence the phrase "started from the bottom now we're here"

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Recursion semantics

$$\mu f = \bigvee_n f^n(\bot) \quad \rightsquigarrow \quad [\![\mu X.\varphi(X)]\!] = \bigcup_n [\![\varphi^n(\bot)]\!]$$

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$$[\![\bot]\!] \subseteq [\![\varphi(\bot)]\!] \subseteq [\![\varphi(\varphi(\bot))]\!] \subseteq \ldots \subseteq [\![\varphi^n(\bot)]\!] \subseteq \ldots$$

If the fixed point is at some power n, then there is a finite increasing chain of sets of states which satisfy $\mu X.\varphi(X)$.

"μ is finite looping"

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$$[\![[a][a]\bot]\!] = \{s \in S \mid \forall t \,.\, s \xrightarrow{a} t \implies t \in [\![[a]\bot]\!]\}$$

$= \text{set of states whose } a \text{ transitions go to states with no } a \text{ transitions}$

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Example

What does this express?

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And so on. If a state s is in $[\![\mu X.[a]X]\!]$, then all a paths starting at s are finite.
We can say $TS \models \varphi$ if every initial state s_0 is in $[\![\varphi]\!]$.
Hence $TS \models \mu X.[a]X$ if TS contains no infinite initial a paths.

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Propositional Dynamic Logic

"I've got a proposition for you..."

– Joseph "Proposition Joe" Stewart

Introduction to PDL

Propositional Dynamic Logic is another modal logic. Labels on modalities like $\langle\alpha\rangle$ and $[\alpha]$ represent (non-deterministic) programs, and we read formulas with these modalities as:

$\langle\alpha\rangle\varphi \quad \mapsto$ "**Some** terminating execution of α ends in a state satisfying φ"

$[\alpha]\varphi \quad \mapsto$ "**Every** execution of α leads to a state satisfying φ"

*

Introduction to PDL

If we give ourselves a set of basic atomic programs $a, b, \ldots$ which go from state to state, we can write more complex programs with the familiar operations in regular expressions.

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$\alpha; \beta$: Sequentially execute α, then β

α^* : Execute α some finite number of times (perhaps 0)

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Syntax of PDL

Formulas in PDL follow the usual syntax

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We can obtain $\langle\alpha\rangle\varphi$ and $\varphi \lor \psi$ by taking the de Morgan dual as usual.
Formulas express properties of states in transition systems, so we may make judgements such as $s \models \varphi$ for some state s, and extend the satisfaction relation to transition systems, such that $TS \models \varphi$ if every initial state $s_0 \models \varphi$.

$*$

Small Model Property

PDL (like the other logics mentioned earlier) has the **small model property**, which means that if φ is satisfiable, i.e. if there is a transition system TS such that $TS \models \varphi$, then there is a finite transition system TS_{FIN} such that $TS_{FIN} \models \varphi$.

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In this way, we get a usable method to transform transition systems satisfying φ into other, finite transition systems.

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There are at most $2^{|\Gamma|}$ equivalence classes in $S/\sim$ (2 possible truth values for each sub-formula); if we let $[s], [t] \in S/\sim$ represent states in a new TS_{FIN}, with $[s] \xrightarrow{a} [t]$ if for some $s' \in [s]$ and $t' \in [t]$, $s' \xrightarrow{a} t'$, then one can show TS_{FIN} also satisfies φ.

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Expressing PDL in $L\mu$

"I'm expressin' with my full capabilities"

– Dr. Dre

Expressing the modalities

Expressing PDL with the tools available in $L\mu$ is simple - the syntax and semantics are similar, with the exception of the ways in which we may combine programs in PDL. The translations for these are still straightforward:

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Verifying these formulas are equivalent is an exercise in semantics; let's look at the most interesting case:

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Using our iteration again, $[\![\varphi \vee \langle \alpha \rangle \bot]\!]$ is the set of all states satisfying φ (no states satisfy $\langle \alpha \rangle \bot$). Then $[\![\varphi \vee \langle \alpha \rangle (\varphi \vee \langle \alpha \rangle \bot)]\!]$ is the set of all states which either satisfy φ, or in which there is a α transition to a state satisfying φ.

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Iterating this, $s \models \mu X.\varphi \vee \langle \alpha \rangle X$ if and only if there is an α path from s reaching a state satisfying φ. This is precisely the condition defining $\langle \alpha^* \rangle \varphi$.

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Showing $L\mu$ is strictly more expressive

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Suppose φ is a PDL formula which is equivalent to $\mu X.[a]X$. Then if $TS \models \mu X.[a]X$, $TS \models \varphi$ as well.

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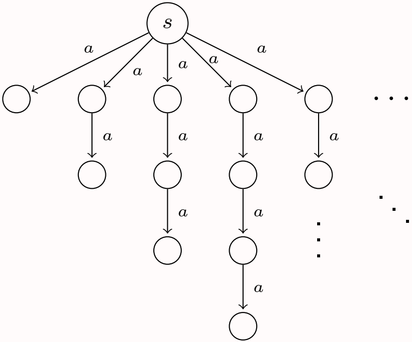
Showing $L\mu$ is strictly more expressive

Consider the following transition system TS with initial state s:



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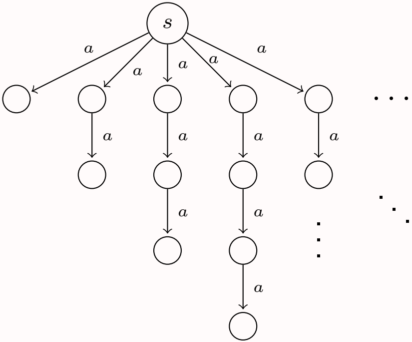

Every path from s is finite length, hence $TS \models \mu X.[a]X$.

$*$

Showing $L\mu$ is strictly more expressive

If φ (the PDL formula) is equivalent to $\mu X.[a]X$, then $TS \models \varphi$ as well.

By the proof of the small model property, we can then collapse TS to a finite TS_{FIN} which also satisfies φ. Since $\varphi \equiv \mu X.[a]X$, it follows that $TS_{FIN} \models \mu X.[a]X$.

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But TS_{FIN} must contain a loop as a result of the filtration process, so there is an infinite a path. This gives a contradiction.

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Showing $L\mu$ is strictly more expressive

So there is no PDL formula equivalent to $\mu X.[a]X$, and $L\mu$ is strictly more expressive than PDL.

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Thank you for your time! Questions?

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